

Long range Ising model for credit risk modeling in homogeneous portfolios

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Within the framework of maximum entropy principle we show that the finite-size long-range Ising model is the adequate model for the description of homogeneous credit portfolios and the computation of credit risk when default correlations between the borrowers are included. The exact analysis of the model suggest that when the correlation increases a first-order-like transition may occur inducing a sudden risk increase. Such a feature is not reproduced by the standard models used in credit risk modeling.

The modelization and control of risk is a subject of extreme importance in a rich variety of disciplines ranging from geology to medicine. To know which is the probability distribution of rare events that may cause very important losses is the key point to be solved and to which a lot of efforts have been devoted for many years. Within the framework of finance, the most studied case is that of market risk (the risk associated to uncertainties of capital markets). But, by far, the risk that most concern practitioners (both due to its negative impact and its modeling difficulties) is credit risk [1]: a bank or financial institution give credits to N borrowers which may fail to return it. Such a group of borrowers constitutes a so called credit portfolio. It is very important to model which is the probability $p(L)$ that L ($0 \leq L \leq N$) of such borrowers will fail in returning the credit. The knowledge of $p(L)$ is of extreme importance to decide the total amount of capital and reserves that the bank should hold to prevent a failure and for economic and regulatory reasons. Also $p(L)$ is the main driver in the valuation and hedging of credit derivatives.

Banks have been using many alternative models. Most of these, as a first step, consider the case of homogeneous portfolios for which the properties of the borrowers and their credits are supposed to be the same. In this paper we will also restrict to this homogeneous case. In the financial literature the description of a portfolio starts by defining the “losses” l_i ($i = 1, \dots, N$) which takes values 0 and 1 depending on whether the borrower i returns the credit or fails to return it. For convenience, we define $S_i = 1 - 2l_i$ which takes values $+1$ when a borrower returns the credit and -1 when it fails to return it. Note that the number of losses is $L = \sum_i l_i = (N - \sum_i S_i)/2$.

The simplest model is the one of independent borrowers. This is characterized by a single parameter which is the probability of default p . In this case, the loss probability is given by the binomial distribution:

$$p_B(L) = \binom{N}{L} (1-p)^{N-L} p^L. \quad (1)$$

The expected value of the number of losses is $\langle L \rangle = N\langle l_i \rangle = Np$ and its variance is $V(L) = N^2 V(l_i) = N^2 p(1-p)$. (The corresponding expressions in terms of the S_i variables can be easily derived taking into account that $\langle S_i \rangle = N(1-2p)$ and $V(S_i) = 4p(1-p)$.) The main drawback of this model is that it does not contain the possibility for correlations between borrowers which are known to exist and play an important role. Although in some cases the links between borrowers can be difficult to understand, in many cases borrowers tend to default in a cooperative manner.

The definition of the default correlation ρ in terms of the S_i variables is given by:

$$\rho = \frac{\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle}{\sqrt{\langle S_i^2 \rangle - \langle S_i \rangle^2} \sqrt{\langle S_j^2 \rangle - \langle S_j \rangle^2}} \quad (i \neq j). \quad (2)$$

The same formula applies for the l_i variables, as can be checked straightforwardly. To include correlations different theoretical models have been proposed by modifying the model of independent borrowers including phenomenological hypothesis on how p depends on external factors [2]. For instance, the economic conditions can be described by a latent variable y which is a random variable with a certain probability density $f(y)$. Then, the concept of default probability p is substituted by the concept of conditional default probability $\phi(y)$ such that for a fixed value of y the probability of L defaults is binomial with $p = \phi(y)$. Therefore,

$$p(L) = \int_{-\infty}^{\infty} dy f(y) \binom{N}{L} [1 - \phi(y)]^{N-L} \phi(y)^L. \quad (3)$$

Such conditionally independent models (CIM) are typically characterized by two parameters which could be empirically determined. First, the average (also called unconditional) default probability given by:

$$\bar{p} \equiv \langle l_i \rangle = \int_{-\infty}^{\infty} dy \phi(y) f(y). \quad (4)$$

Second, the default correlation which, for the case of homogeneous portfolios, is given by:

$$\rho = \frac{\int_{-\infty}^{\infty} \phi(y)^2 f(y) dy - \left[\int_{-\infty}^{\infty} \phi(y) f(y) dy \right]^2}{\int_{-\infty}^{\infty} \phi(y) f(y) dy - \left[\int_{-\infty}^{\infty} \phi(y) f(y) dy \right]^2}. \quad (5)$$

For instance, the Merton based model (used by JPMorgan and KMV companies in their credit portfolio models) corresponds to the assumption that $f(y)$ is the $N(0, 1)$ Gaussian probability density and the function ϕ is:

$$\phi_M(y) = \Phi\left(\frac{c - \sqrt{q}y}{\sqrt{1-q}}\right) \equiv \int_{-\infty}^{\frac{c - \sqrt{q}y}{\sqrt{1-q}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz, \quad (6)$$

where Φ is the error function, c is the “critical value” and q is the “asset correlation” (which should not be confused with the default correlation). For this case it is easy to check that the average default probability is $\bar{p} = \Phi(c)$, and that the default correlation is:

$$\rho = \int_{-\infty}^c \int_{-\infty}^c \frac{du dv}{2\pi\sqrt{1-q^2}} e^{-(u^2+v^2-2quv)/2(1-q^2)}. \quad (7)$$

Other examples of CIM are the CreditPortfolioView model (of McKinsey) and the CreditRisk+ model (of Credit Suisse Financial Products) which use different functions $\phi(y)$ and $f(y)$ [3, 4]. The CIM have been shown to only capture cyclic correlations, leaving aside direct correlations, contagion and cascade effects [5].

In this letter we propose a different method to model a credit portfolio that includes default correlations in a fundamentally different way. We describe portfolios with vectors $X = (S_1, S_2, \dots, S_N)$. The set of all possible 2^N vectors X is, therefore, our phase space $\Omega = \{X\}$. To solve a credit portfolio model is to assign a probability law $P(X)$ on Ω . Instead of trying to derive which are the dependencies between the different borrowers or between the borrowers and the external factors, we will assume that default correlation ρ is different from zero and derive $P(X)$ from the Maximum Statistical Entropy principle. This principle, which is essential in Information Theory, has been widely used for building probability distributions when a underlying theory is lacking. It states that one should consider the model $P(X)$ that maximizes the entropy functional:

$$S[P] = \sum_{X \in \Omega} P(X) \ln P(X), \quad (8)$$

subject to the conditions imposed by the previous known information. Such a method allows to control what are the exact ingredients (or the exact information) that is taken into account to formulate the model. This is opposite to the above presented theoretical models which assume a rigid structure for relations between borrowers that cannot be empirically checked.

It is instructive to first present the derivation of the simplest model. We will impose only two conditions. First the normalization condition:

$$\sum_{X \in \Omega} P(X) = 1. \quad (9)$$

Second we assume that we have a previous knowledge of the expected value of the number of losses $\langle L \rangle$ or, equivalently, of the average default probability \bar{p} . In terms of the probability law $P(X)$ this condition is written as:

$$\langle \sum S_i \rangle = \sum_{X \in \Omega} P(X) M(X) = N(1 - 2\bar{p}), \quad (10)$$

where we have defined the function $M(X) = \sum_i S_i$. We want to stress that the knowledge of the average default probability \bar{p} allows to fix the expected value of M but not its exact value. Maximize (8) with conditions (9) and (10) is equivalent to maximize the Lagrange functional:

$$\begin{aligned} \Psi_0[P] = & \sum_{X \in \Omega} P(X) \ln P(X) - \lambda \left(\sum_{X \in \Omega} P(X) - 1 \right) \\ & - \alpha \left(\sum_{X \in \Omega} M(X) P(X) - N(1 - 2\bar{p}) \right), \end{aligned} \quad (11)$$

where λ and α are Lagrange multipliers. The result of the maximization is $P(X) = e^{\lambda - 1 + \alpha M(X)}$. The parameter λ can be easily deduced imposing (9). To obtain α we require $\langle M(X) \rangle = N(1 - 2\bar{p})$. A straightforward algebra renders $\alpha = \tanh^{-1}(1 - 2\bar{p})$. The probability $P(X)$ results in:

$$P(X) = (1 - \bar{p})^{\frac{N+M(X)}{2}} \bar{p}^{\frac{N-M(X)}{2}}. \quad (12)$$

Note that this probability depends on X only through the function $M(X)$. Since there are $\binom{N}{L}$ different vectors that give the same loss L , the probability $p(L)$ turns out to be Eq. (1) substituting p by \bar{p} .

The inclusion of the existence of correlations between borrowers into the model requires to introduce a third condition in the maximization of (8). In this case, the symmetric function of X whose expected value is fixed by the default correlation ρ is:

$$H(X) = \frac{2}{N-1} \sum_{i>j} S_i S_j. \quad (13)$$

Note that the number of terms in the sum is $N(N-1)/2$ so that the pre-factor $2/(N-1)$ ensures that $H(X) \propto N$. Using (2), (10) and the fact that $\langle S_i^2 \rangle = 1$ it is straightforward to check that the knowledge of ρ allows to fix the average value of $\langle H(X) \rangle$ according to:

$$\langle H(X) \rangle = N [1 - 4\bar{p}(1 - \bar{p})(1 - \rho)]. \quad (14)$$

Imposing this condition, together with conditions (9) and (10), leads to the following Lagrange functional:

$$\Psi[P] = \Psi_0[P] - \beta \left[\sum_{X \in \Omega} H(X)P(X) - \langle H(X) \rangle \right]. \quad (15)$$

Differentiating and imposing (9) to eliminate λ one obtains:

$$P(X) = \frac{e^{\alpha M(X) + \beta H(X)}}{\sum_{X \in \Omega} e^{\alpha M(X) + \beta H(X)}}. \quad (16)$$

This can be rewritten as $P(X) = e^{-\mathcal{H}} / \sum_{X \in \Omega} e^{-\mathcal{H}}$ with:

$$\mathcal{H}(X) = -\frac{2\beta}{N-1} \sum_{i < j} S_i S_j - \alpha \sum_{i=1}^N S_i, \quad (17)$$

which is the canonical probability distribution corresponding to the well known long-range Ising model (LRIM) of a finite system with N spins and a convenient redefinition of the exchange constant and the external field [6]. Note that $P(X)$ in (16), again, is only a function of $M(X)$ since $H(X) = \frac{1}{N-1} (M(X)^2 - N)$. Therefore the probability of L losses will be obtained by multiplying expression (16) by the binomial number $\binom{N}{L}$.

To determine the values of α and β in (16) that allow to impose conditions (10) and (14) for given values of \bar{p} and ρ is a difficult task that has been extensively studied in the field of Statistical Mechanics. Note that the problem when $N \rightarrow \infty$ can be easily solved by a saddle point method. This method renders a solution that coincides with the solution obtained by the mean field hypothesis [7]. But, such an approximation is of little interest here because of two reasons: credit portfolios, although being large, are quite far from the thermodynamic limit and such a mean field approximation immediately leads to $\rho = 0$ (except for the unrealistic case of $\bar{p} = 0.5$).

Fig. 1 shows an example of the probability $p(L)$ derived by the LRIM corresponding to $\bar{p} = 0.02$ and $\rho = 0.01$ ($\alpha = 8.85 \cdot 10^{-3}$, $\beta = 1.01148$). The behaviour is compared with the binomial model (without correlation) and the Merton based model. In this last case the parameters ($c = -2.054$ and $q = 0.0718$) have been calibrated in order to have the same values of \bar{p} and $\rho = 0.01$. Note that the LRIM is the only model able to capture the existence of a second peak in the large L region (negative M region) that appears due to the existence of correlations which favour a collective behaviour of the borrowers.

Although the peak in the right (large L region) is usually very small compared to the peak in the left (small L region) it has strong implications in the evaluation of credit risk. There are different strategies to measure risk. A common one is the computation of the so called Value-at-Risk (VaR): this is the number of failures Λ that must be considered in order to ensure that the probability of

having more than Λ fails is smaller than a certain tiny confidence level α_c . Thus Λ is given by the solution of the equation:

$$\sum_{L=\Lambda}^N p(L) = \alpha_c \quad (18)$$

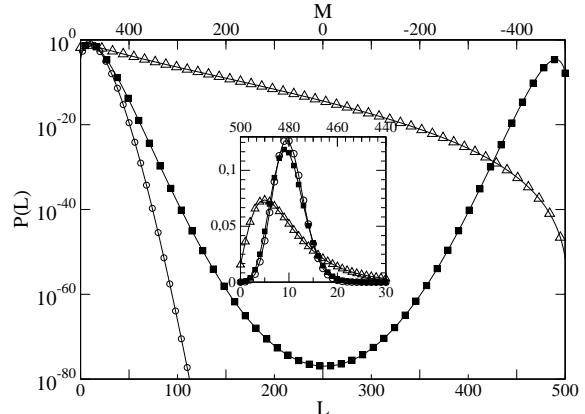


FIG. 1: Comparison of the probability of L failures $p(L)$ (in linear-log scale) obtained from the LRIM (\blacksquare), the binomial model (\circ) and the Merton based model (\triangle) with $N = 500$ and $\bar{p} = 0.02$. Both the LRIM and the Merton model correspond to $\rho = 0.01$. The inset shows the peak on the left in linear scale. The scales on top show the corresponding values of M .

The smaller the confidence level, the larger Λ . Fig. 2 shows an example of the behaviour of the sum in the left of (18) as a function of a variable Λ . The horizontal line indicates the confidence level. Note that given the two-peaked shape of $p(L)$, the sum in (18) exhibits a constant plateau. Consequently a small change in ρ may produce a drastic change in the position of the VaR (crossing point).

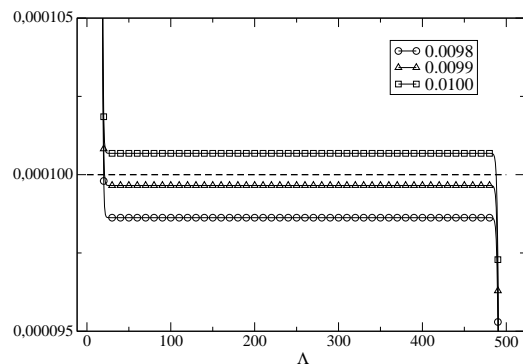


FIG. 2: Calculation of the VaR for a given confidence level $\alpha_c = 0.0001$ (dashed line). The continuous lines with symbols show the behaviour of the left hand side in equation 18 as a function of Λ for the LRIM with $\bar{p} = 0.01$ and different values of ρ , as indicated by the legend.

From the physical point of view, this change is very similar to a first-order phase transition in which the order parameter (VaR) exhibits a discontinuity. The main difference is that our system is small and therefore real discontinuities do not exist but, instead, there are sharp changes. In order to make clear this sharp behaviour we show in Figure 3 the VaR as a function of ρ for $\bar{p} = 0.01$ and $\alpha_c = 0.0001$. The Merton model displays a smooth increase of the VaR with increasing correlation. For small correlations the differences between the LRIM and the binomial one are negligible. However, there exist a value of the correlations for which, according to the LRIM, a dramatic increase in the VaR occurs, which is not reproduced by the Merton model.

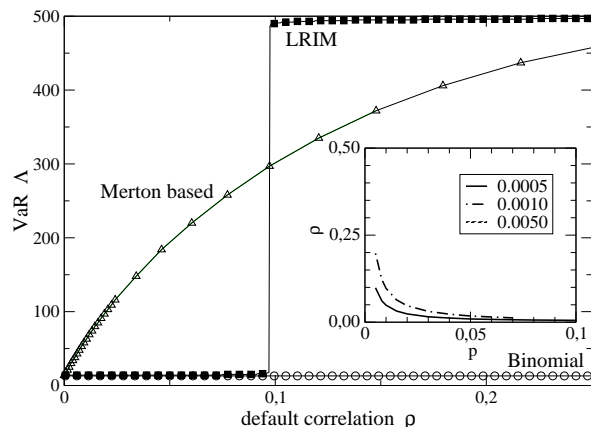


FIG. 3: VaR as a function of the default correlation ρ for the LRIM, the Merton model and the binomial model (independent of ρ). The three models correspond to $N = 500$ and $\bar{p} = 0.01$. The inset shows the position of the sharp increases of the VaR in the \bar{p} - ρ space predicted by the LRIM, for different values of the confidence level α_c , as indicated by the legend.

It is of paramount importance to locate the region where such strong change in the VaR occurs for different values of the confidence level α_c . This is shown in the inset of Fig. 3. The linear axes represent the space of empirical parameters \bar{p} and ρ . The different lines, corresponding to different confidence levels, separate the region in which the behaviour of the LRIM is similar to the binomial model (below) from the region where collective failure may occur and thus predictions with the binomial model strongly underestimate the VaR. Other measurements of risk (like the expected shortfall) may exhibit even stronger differences.

There are a series of possible extensions of the LRIM model: first, the extension to the case of non-homogeneous portfolios is straightforward. The borrowers need to be separated in different tranches. For each tranche the formulation of the model will require the knowledge of the default probability, the autocorrelation within each tranche and the correlations among different

tranches. Note that in the LRIM model will be easy to introduce negative correlations between certain tranches to represent big competitors in small markets. This is impossible to reproduce with the conditional independent models. A second interesting extension will be to change the discrete variables S_i to continuous variables ϕ_i . The Ising model, then, be substituted by a ϕ^4 model. This will enable to describe the fact that when a borrower defaults there is always a recovery of part of the loan. A third extension could be the inclusion of randomness in the LRIM. For instance random fields and/or random bonds would allow to take into account the uncertainties in the knowledge of the empirical values of ρ and \bar{p} . Finally one could include higher order interaction terms, like three spin terms. This would allow to introduce, in a controlled way, the existence of three borrower correlations which are known to exist when loans are fully guaranteed by third parties. Note that the conditional independent models do also introduce higher order correlations, but in a completely uncontrolled manner.

Within the framework of information theory, it has been shown that the LRIM is the natural model to be used for the description of credit portfolios and risk modelling when knowledge about the existence of correlations is taken into account. One of the main results is that risk measurements can display big and sharp increases due to default correlations, in regions of the space parameters where other models fail to predict them. The discussion presented also shows that the LRIM can be useful not only in finance but, in general, for any complex system described by binary variables for which the available information is the average value and the correlations.

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